ADVANCED MATHEMATICS

Final Exam - January 2013

Name: ____________________

NIU: ____________________ Group: ______

Grade:

Instructions: The exam consists of six questions. You have two hours to give a reasoned answer to all the exercises. Write the quiz entirely in ink.

1 Determine for which values of the parameter a the matrix A is diagonalizable.

$$A = \begin{pmatrix} 0 & 0 & a \\ 2 & 1 & 2 \\ a & 0 & 0 \end{pmatrix}$$

2 Suppose that in a given market with a single commodity the demand function is $D(P) = 2 - P$ and the supply function is $S(P) = -1 + 2P$, where $P > 0$ denotes the unitary price of the good. Assume that time is discrete and that the market follows the dynamics of the Cobweb Model, that is, $S(P_t) = D(P_{t+1})$ for every t. Calculate the equilibrium price and study whether it is globally asymptotically stable.

3 Solve the system of difference equations

$$X_{t+1} = AX_t,$$

where A is the matrix of Exercise 1 when $a = 2$. Is this system globally asymptotically stable? Is there any initial condition X_0 such that the solution converges?

4 Solve the following differential equation:

$$x' = -\frac{x^2 + x + 2tx}{t^2 + t + 2tx},$$

where $x(0) = 1$.

5 Obtain the solutions of the following differential equation:

$$x''' - 3x'' + 3x' - x = \sin t$$

$\boxed{6}$ Obtain the solution to the following system:

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X + \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix},$$

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SOLUTIONS

Instructions: The exam consists of six questions. You have two hours to give a reasoned answer to all the exercises. Write the quiz entirely in ink.

1 Determine for which values of the parameter a the matrix A is diagonalizable.

$$A = \begin{pmatrix} 0 & 0 & a \\ 2 & 1 & 2 \\ a & 0 & 0 \end{pmatrix}$$

Solution. The eigenvalues of A are $\sigma(A) = \{1, a, -a\}$. If $a \in \mathbb{R} \setminus \{0, 1, -1\}$ then the three roots are real and different each other, and A is diagonalizable. We study the other three cases:

(a) If $a = 0$. Then $\sigma(A) = \{1, 0\}$ with $m(1) = 1$ and $m(0) = 2$. We compute the dimension of $S(0)$, $\dim S(0) = 3 - \text{rank}(A - 0 \cdot I_3) = 3 - 1 = 2$, which coincides with its multiplicity. Therefore, the matrix A is diagonalizable.

(b) If $a = -1$. Then $\sigma(A) = \{1, -1\}$ with $m(-1) = 1$ and $m(1) = 2$. We compute the dimension of $S(1)$, $\dim S(1) = 3 - \text{rank}(A - 1 \cdot I_3) = 3 - 1 = 2$, which coincides with its multiplicity. Therefore, the matrix A is diagonalizable.

(c) If $a = 1$. Then $\sigma(A) = \{1, -1\}$ with $m(1) = 2$ and $m(-1) = 1$. On the other hand, $\dim S(1) = 3 - \text{rank}(A - 1 \cdot I_3) = 3 - 2 = 1$. Since the dimension of the eigenspace and the multiplicity do not coincide, the matrix A is not diagonalizable.

2 Suppose that in a given market with a single commodity the demand function is $D(P) = 2 - P$ and the supply function is $S(P) = -1 + 2P$, where $P > 0$ denotes the unitary price of the good. Assume that time is discrete and that the market follows the dynamics of the Cobweb Model, that is, $S(P_t) = D(P_{t+1})$ for every t. Calculate the equilibrium price and study whether it is globally asymptotically stable.

Solution. The difference equation we need to solve is

$$-1 + 2P_t = 2 - P_{t+1} \quad \equiv \quad P_{t+1} + 2P_t = 3$$

The characteristic polynomial of the associated homogeneous equation is $r + 2 = 0$; hence $x_t^h = A_0 (-2)^t$. Since the independent term $b_t = 3$ is a polynomial of degree zero, we try as particular solution $x_t = C$. Once we substitute into the equation we obtain that $x_t^p = C = 1$. Finally

$$p_t = A_0 (-2)^t + 1$$

The equilibrium price is $p = 1$, and it is not globally asymptotically stable.

3 Solve the system of difference equations

$$X_{t+1} = AX_t,$$

where A is the matrix of Exercise 1 when $a = 2$. Is this system globally asymptotically stable? Is there any initial condition X_0 such that the solution converges?

Solution. When $a = 2$, $\sigma(A) = \{1, 2, -2\}$, and $S(1) =< (0, 1, 0) >$, $S(2) =< (1, 4, 1) >$ and $S(-2) =< (-1, 0, 1) >$. The system is homogeneous, and therefore

$$X_t = A_0 1^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + A_1 2^t \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + A_2 (-2)^t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Since not all the eigenvalues are smaller than 1, the system is not globally asymptotically stable. On the other hand, the solution will converge for an initial condition X_0 only if $A_1 = A_2 = 0$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = X_0 = A_0 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \Leftrightarrow \quad \alpha = \gamma = 0.$$

For $X_0 = (0, 1, 0)$, for example, the solution converges.

4 Solve the following differential equation:

$$x' = -\frac{x^2 + x + 2tx}{t^2 + t + 2tx},$$

where $x(0) = 1$.

Solution. It is an exact equation whose canonical form is $(x^2 + x + 2tx)dt + (t^2 + t + 2tx)dx = 0$ In this case, $P(t, x) = x^2 + x + 2tx$ and $Q(t, x) = t^2 + t + 2tx$. We check the condition to be exact:

$$\frac{\partial P}{\partial x} = 2x + 1 + 2t = \frac{\partial Q}{\partial t}.$$

- Let $F(t, x)$ be the solution of the equation we are looking for.
- Impose that
$$\frac{\partial F(t, x)}{\partial t} = P(t, x) \Rightarrow \frac{\partial F(t, x)}{\partial t} = x^2 + x + 2tx,$$
by integrating both sides with respect to t, we can isolate $F(t, x)$.
$$F(t, x) = x^2t + xt + xt^2 + h(x).$$
- Now, impose that
$$\frac{\partial F(t, x)}{\partial x} = Q(t, x) \;\Leftrightarrow\; h'(x) = 0.$$
- To obtain $h(x)$, simply integrate.
$$h(x) = 0.$$
- Substitute $h(x)$ in the expression of Step 2, and then:
$$F(t, x) = x^2t + xt + xt^2.$$
- The solution to the exact equation is given in implicit form:
$$x^2t + xt + xt^2 = C$$
Now, we impose the initial condition $x(0) = 1$ to obtain that $C = 0$. Therefore, the solution is $t^2 + x^2 - 3tx = 0$.

5 Obtain the solutions of the following differential equation:

$$x''' - 3x'' + 3x' - x = \sin t$$

Solution. The roots of the characteristic polynomial are $r = 1$ with $m(1) = 3$. Then,

$$x^h(t) = A_0e^t + A_1te^t + A_2t^2e^t$$

Since $b(t) = \sin t$, we propose $x(t) = C \sin t + D \cos t$. Taking the needed derivatives and substituting we get that $C = \frac{1}{4}$ and $D = \frac{-1}{4}$. And the,

$$x^p(t) = \frac{1}{4}\sin t - \frac{1}{4}\cos t$$

Finally

$$x(t) = A_0e^t + A_1te^t + A_2t^2e^t + \frac{1}{4}\sin t - \frac{1}{4}\cos t$$

6 Obtain the solution to the following system:

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X + \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix},$$

Solution. Since the matrix is already diagonal, then we know that $D = P = P^{-1} = I_3$, which implies that the solution to the associated homogeneous system is:

$$X^h(t) = K_0 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_1 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In order to obtain a particular solution, we solve the system

$$K_0' e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_1' e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix} \quad \equiv \quad K_0' = K_1' = 1 \quad \equiv \quad K_0 = K_1 = t$$

Then, a particular solution is

$$X^p(t) = te^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} te^t \\ te^{2t} \end{pmatrix}$$

Finally,

$$X(t) = K_0 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_1 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} te^t \\ te^{2t} \end{pmatrix}$$